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On the generation of spatially growing waves in a boundary layer

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The solution is obtained in general terms for the velocity fluctuations generated in a laminar boundary layer by an oscillating disturbance on the boundary wall. The form of excitation is chosen to represent a vibrating ribbon of the type used by Schubauer to force disturbances in boundary layers. The forced wave system generated by the ribbon is shown to be a spatially growing one, which is described far downstream by an eigenmode of the system which has a complex wave-number.

1. Introduction

The theoretical study of hydrodynamic stability deals solely with the behaviour of unexcited free modes. The detailed mechanisms of the initial creation of these modes are not considered. In boundary layers, for example, travelling wave disturbances have been observed in the transition zone, and although stability theory attempts to describe their growth no effort has so far been made to discuss the process of their generation. The origin of these waves is rather loosely attributed to such external factors as noise, free-stream turbulence and wall roughness. To understand more about transition it is necessary to investigate these initial stages in the formation of travelling waves, and in this paper an attempt will be made to study certain simple forced oscillations and show that a disturbance like Schubauer's ribbon excites a spatially growing wave.

Much of the detailed experimental work on stability has necessarily been carried out on some form of forced oscillation and it is therefore important to be able to relate these observed driven modes with those free waves theoretically discussed. In their now classic experiment Schubauer & Skramstad (1947) used a vibrating ribbon to excite waves in a boundary layer on a flat plate. The behaviour for free modes calculated by Schlichting (1933, 1935) was followed reasonably well in most respects by these forced oscillations. However, in the experiment the steadily vibrating ribbon quite naturally generated a train of waves which grew (or decayed) spatially as they propagated away from the source, while the theoretical treatment discussed a wave system which grew in amplitude with respect to time. Schubauer compared the theoretically predicted temporal growth rates with those apparent to an observer moving with the wave system in the experiment. However, if there is any dispersion there is no justification for this procedure, and it is necessary to transform from temporal to spatial growth using the group rather than the phase velocity. Schlichting did in fact choose to use the group velocity to obtain rates of spatial growth, but he offered no justification for this step. A proof of this relationship between the two amplification parameters has been given (Gaster 1962) for cases where the amplification rates are small, as in boundary layer and other similar flows. Now, in the case of the flat-plate boundary layer both experiment and theory have shown that the wave speed is almost independent of wave-number. Thus the group and phase velocities are roughly equal and a reasonable agreement between theory and experiment was achieved by Schubauer.

Attempts to examine the spatial growth of linear disturbances have been made by Criminale & Kovasnay (1962) and Brooke Benjamin (1961), who examined the behaviour of pulses of waves propagating downstream. The individual waves were temporally growing, but by following the wave packet, which travels at the group velocity, a spatial growth can be derived. In fact, for the small rates of amplification arising in boundary-layer flows this procedure will lead to the correct result because of the special relationship between the two rates of growth. But in general the integrals involved in the analysis should be carried out on some contour in the complex wave-number plane in order to provide the best form of asymptotic expansion. If this is done the wave packet will be composed of modes which are exponential in both space and time.

It will be shown that the experimentally observed type of wave can be treated theoretically by considering a class of possible solutions to the Orr–Sommerfeld equation that are spatially exponential. Spatially growing waves of finite amplitude have been studied in plane Poiseuille flow by Bradshaw, Stuart & Watson (1960) and Watson (1962). It is argued by Lin (1955) that solutions of this form are not allowed in a linear theory where the amplitude must be bounded for all x. Although it is true that any spatially growing linear theory must fail sufficiently far from the source, it is not unreasonable to apply such a theory to some region where non-linear effects are small.

In order to be able to solve the linearized perturbation equations of motion it is necessary to use the parallel mean flow approximation to simplify the problem as much as possible. That is, the section of boundary layer under investigation is considered to be sufficiently parallel to allow one to neglect terms arising from variations of mean flow with x. This approximation has been justified by Pretsch (1941) who showed that for neutral modes in the zero-pressure-gradient case the additional terms are negligible, particularly at high Reynolds numbers. In discussions on boundary-layer stability it is usual to consider an infinite parallel flow having a uniform mean velocity profile given by that of the real flow at some station of interest. This infinite flow field model appears to confine the possible modes of instability to those periodic in x, but there seems to be no reason why such a restriction should apply to the real boundary-layer problem which is not of infinite extent. The parallel flow treatment essentially offers a simplification to the equations of motion over some region of the flow, and in the following analysis we will consider the behaviour of forced types of disturbance within this region. Forcing will be provided by a modification to one of the boundary conditions at the wall to simulate roughly the effects of a vibrating ribbon, which has so often been employed in experiments on instability waves (Schubauer & Skramstad 1947; Liepmann 1943, 1945; Klebanoff & Tidstrom 1959 and Klebanoff, Tidstrom & Sargent 1962). It will be shown in the following analysis that the group velocity is the proper parameter deciding the direction of disturbance propagation, and its sign is thus of vital importance when discussing wave growth. In solutions of the Orr–Sommerfeld equation for boundary layers and other similar almost parallel flows which have velocity profiles without any reversed flow, it appears that the group velocity is always positive, although there is no *a priori* reason why this should be so. We will consider both positive and negative values of the group velocity, although the latter case may have no physical significance in the types of flow usually considered.

2. Formulation of the problem

The behaviour of small-amplitude disturbances in a given parallel mean flow is governed by a set of equations which become homogeneous after linearization with respect to the disturbance amplitude. As the boundary conditions are also homogeneous the problem is a characteristic-value one for the determination of the possible eigenmodes. A disturbance can be forced either by a modification to the equations by the addition of some body-force terms, or by a change in the boundary conditions so that they are no longer homogeneous. In this paper we will discuss the effect of altering one of the boundary conditions to simulate an excitation like Schubauer's ribbon.

The boundary conditions to be imposed on the perturbation stream function are:

at the wall

$$y = 0, \quad u = \partial\psi(0;x,t)/\partial y = 0, \quad v = -\partial\psi(0;x,t)/\partial x = Q(x,t),$$

where $Q(x,t)$ is some specified forcing function,

far from the wall, at y_1, say, the perturbation must decay, i.e.

$$u = \partial\psi(y_1;\, x,t)/\partial y \to 0 \quad \text{and} \quad v = -\partial\psi(y_1;\, x,t)/\partial x \to 0 \quad \text{as} \quad y_1 \to \infty.$$

A form of $Q(x,t)$ roughly simulating a ribbon vibrating near the surface is $\delta(x)\cos\omega t$. However, in attempting to evaluate $\psi(y;\, x,t)$ for this disturbance it is not possible to decide which is the correct integration contour so that more than one solution appears to be possible. This difficulty commonly arises in wave problems but usually one can reject the spurious solution by physical reasoning. For example, one may reject a solution which enables energy to propagate towards the source. Arguments of this type are avoided in the present analysis by considering the disturbance

$$\begin{aligned} Q(x,t) &= \delta(x)\cos\omega t \quad (t>0), \\ &= 0 \qquad\qquad\quad (t<0), \end{aligned}$$

and finding the asymptotic solution for large t. Any ambiguity in the contour to be taken can then be avoided by applying the initial condition that no perturbation exists anywhere in the flow prior to time $t = 0$.

We can define the perturbation stream function of the disturbance by

$$\psi(y;\, x,t) \equiv c_1\int_{-\infty}^{+\infty} c_2\int_{-\infty}^{+\infty} \Phi(y;\, \alpha,\beta)\, e^{i(\alpha x-\beta t)}\, d\alpha\, d\beta, \tag{1}$$

where the integrals are evaluated along contours c_1 and c_2 in the α and β planes respectively. If we assume that Φ is continuous and has continuous derivatives along c_1 and c_2, (1) can be substituted into the linearized equations of motion for parallel mean flows to yield the Orr–Sommerfeld equation,

$$(U(y)-\beta/\alpha)(\Phi''-\alpha^2\Phi)-U''(y)\,\Phi = (-i/\alpha R)(\Phi^{\text{iv}}-2\alpha^2\Phi''+\alpha^4\Phi), \qquad (2)$$

where $U(y)$ is the mean flow, R the Reynolds number and primes denote differentiation with respect to y. At this stage in the analysis both the wave number α and the frequency β are considered to be generally complex parameters.

The fourth-order equation in Φ has four fundamental solutions ϕ_1, ϕ_2, ϕ_3 and ϕ_4 so that the general solution is

$$\Phi(y) = A\phi_1+B\phi_2+C\phi_3+D\phi_4. \qquad (3)$$

Lin has pointed out that each of these fundamental solutions must be entire functions of y because all of the coefficients in (3) are regular everywhere in the finite part of the complex y-plane. He further remarked that these solutions are also entire functions of the parameters α and β in the same domain. However, some care is needed here, for not all forms of possible solutions ϕ_1, etc., are necessarily without branch points, although by combining these in various ways one can always obtain four fundamental functions which are single valued. In the following work we shall assume that the forms of solution selected are entire functions of both α and β. To retain this property over the whole of the α and β planes it is necessary to restrict the analysis to problems of flows in parallel strips, the boundary-layer problem being considered as the limiting case where the upper boundary tends to infinity. This restriction is imposed because the determinantal equations for the constants A, B, C and D contain some elements which tend to infinity with y and other ones which tend to zero. It is therefore necessary to expand the determinants before taking the limit as the upper boundary goes to infinity.

In terms of Φ the boundary conditions are

$$\Phi(y_1;\,\alpha,\beta)\to 0, \quad \Phi'(y_1;\,\alpha,\beta)\to 0 \quad \text{as} \quad y_1\to\infty,$$

and

$$\Phi'(0;\,\alpha,\beta) = 0 \quad \text{at} \quad y=0.$$

The remaining boundary value on $\Phi(0;\,\alpha,\beta)$ is obtained by inverting equation (1). In general (1) cannot be inverted by Fourier's theorem unless it can be shown that the integrand is analytic within the regions containing the contours c_1 and c_2. For the particular case of obtaining $\Phi(0;\,\alpha,\beta)$ it will later be shown that the integrand is in fact analytic over the whole α-plane and in the upper half of the β-plane so that the inversion formula can be used provided c_2 is in the upper half of the β-plane.

From (1),

$$-\frac{\partial\psi}{\partial x}(0,x,t) = -i c_1\!\int_{-\infty}^{+\infty} c_2\!\int_{-\infty}^{+\infty} \alpha\Phi(0;\,\alpha,\beta)\,e^{i(\alpha x-\beta t)}\,d\alpha\,d\beta.$$

Using Fourier's inversion formula we obtain

$$-i\alpha\Phi(0;\,\alpha,\beta) = \frac{1}{4\pi^2}\int_{-\infty}^{+\infty}\int_{0}^{+\infty}\delta(x)\cos\omega t\,e^{-i(\alpha x-\beta t)}\,dx\,dt \qquad (4)$$

or

$$\Phi(0;\,\alpha,\beta) = -\beta/4\pi^2\alpha(\beta-\omega^2) \qquad (5)$$

provided that β_i (the imaginary part of β) is greater than zero so that the integral converges as t tends to infinity. The same restriction must be imposed when (1) is evaluated in the β-plane; c_2 must lie in the upper half-plane.

From (1) and (5) we obtain

$$v(y;\, x,t) = -\frac{\partial \psi}{\partial x}(y;\, x,t) = \frac{i}{4\pi^2} \int_{c_1\, -\infty}^{+\infty} \int_{c_2\, -\infty}^{+\infty} \frac{\beta}{\beta^2-\omega^2} \frac{\Phi(y;\, \alpha,\beta)}{\Phi(0;\, \alpha,\beta)} e^{i(\alpha x-\beta t)}\, d\alpha\, d\beta. \quad (6)$$

For the special case of $v(0;\, x,t)$ the integrand reduces to $\beta/(\beta^2-\omega^2)$ which is analytic over the whole α-plane and also in the upper half of the β-plane as required for the application of the inversion formula.

Free modes of the system are obtained by equating the input disturbance to zero,

$$\Phi(0;\, \alpha,\beta) = 0, \quad (7)$$

which is the characteristic equation describing all the possible eigenmodes of the Orr-Sommerfeld equation with both α and β as complex parameters.

3. Evaluating the integral

Before evaluating (6) it is necessary to discuss the behaviour of the function $\Phi(y;\, \alpha,\beta)$ over the α and β planes. Since $\Phi(y;\, \alpha,\beta)$ is an entire function of the parameters α and β it follows that the ratio $\Phi(y;\, \alpha,\beta)/\Phi(0;\, \alpha,\beta)$ must be meromorphic, the poles arising from the zeros of $\Phi(0;\, \alpha,\beta)$. $\Phi(0;\, \alpha,\beta)$ is an entire function of α for every value of β and may thus be represented by a polynomial. By factorising this polynomial we can obtain all the solutions of equation (7), the number of solutions being given by the highest power arising in the polynomial. Each one of these solutions gives rise to a pole in the integrand of (6) and so contributes to the perturbation. However, from previous studies of stability problems, where one is interested in the most highly amplified (or least damped) mode, it is found that only one important solution arises and this mode has a positive wave speed ($\beta_r/\alpha_r > 0$). As all other modes are highly damped they will not produce any significant contributions to the overall value of (6) and will therefore be neglected. In addition to this pole on the α-plane which is related to the free mode behaviour, there will in general also be poles along the imaginary axis to determine the transient nature of the solution.

In order to simplify the evaluation of (6) it is convenient to change the range of integration. (6) may be written

$$\begin{aligned} v(y;\, x,t) = {} & \frac{1}{4\pi^2} \int_{-\infty}^{+\infty} \int_{0}^{\infty} \frac{i\beta}{\beta^2-\omega^2} \frac{\Phi(y;\, \alpha,\beta)}{\Phi(0;\, \alpha,\beta)} e^{i(\alpha x-\beta t)}\, d\alpha\, d\beta \\ & + \frac{1}{4\pi^2} \int_{-\infty}^{+\infty} \int_{-\infty}^{0} \frac{i\beta}{\beta^2-\omega^2} \frac{\Phi(y;\, \alpha,\beta)}{\Phi(0;\alpha,\beta)} e^{i(\alpha x-\beta t)}\, d\alpha\, d\beta. \end{aligned}$$

By changing the sign of α and β and rearranging the limits the second integral reduces to

$$\frac{1}{4\pi^2} \int_{-\infty}^{+\infty} \int_{0}^{\infty} \frac{-i\beta}{\beta^2-\omega^2} \frac{\Phi(y;\, -\alpha,\, -\beta)}{\Phi(0;\, -\alpha,\, -\beta)} e^{-i(\alpha x-\beta t)}\, d\alpha\, d\beta.$$

Now on changing the sign of α, and β in the Orr–Sommerfeld equation the complex conjugate equation is formed. Thus $\Phi(y;\,-\alpha,-\beta)$ and $\Phi(y;\alpha,\beta)$ must also be complex conjugates and we can write

$$v(y;\,x,t) = 2\,\mathrm{Re}\left\{\frac{i}{4\pi^2}c_1\int_{-\infty}^{+\infty}c_2\int_0^{\infty}\frac{\beta}{\beta^2-\omega^2}\frac{\Phi(y;\,\alpha,\beta)}{\Phi(0;\alpha,\beta)}e^{i(\alpha x-\beta t)}d\alpha\, d\beta\right\}, \tag{8}$$

where Re{ } denotes the real part of { }.

It remains to discuss the behaviour of $\Phi(y)/\Phi(0)$ for large real values of α and β to show that the integrals converge. We can obtain the asymptotic form of the solution of the Orr–Sommerfeld equation as α_r and β_r tend to infinity. Consider first the case of large positive wave-number, it is clear that the first term of an asymptotic expansion in α_r will consist of terms like $ye^{\pm\alpha_r y}$ and $e^{\pm\alpha_r y}$. Now the perturbation velocities must tend to zero for large y, and therefore taking the negative exponent we see that $\Phi(y)$ must be exponentially small for large α_r for all values of y greater than zero, that is $\Phi(y)/\Phi(0)$ behaves like $e^{-\alpha_r y}$ for large α_r. Also for large negative values of α_r it is clear that $\Phi(y)/\Phi(0)$ decays like $e^{\alpha_r y}$. Convergence over the β-plane can be demonstrated in a similar manner by taking the major terms in the expansion of Φ for large β_r.

Integrating (6) with respect to α we will consider two possible paths of integration, the correct contour being revealed only after carrying out the integration over the β-plane and fitting the initial conditions. First, we will discuss the contributions of a pole away from the imaginary axis, later the effect of a pole on this axis will be deduced. This procedure is permissible since the resulting solution will be made up from the sum of all individual contributions arising from the separate poles in the α-plane.

(*a*) Contour (*a*) is deformed so that it passed from $-\infty$ to $+\infty$ above the pole. For positive values of x the contour may be closed by an infinite semi-circle in the upper half plane, but since the integrand is analytic within this enclosed region the integral is zero.

$$I = 0 \quad \text{for} \quad x > 0,$$

where I is the contribution to $v(y;\,x,t)$ of this pole. For negative x the contour is closed in the lower half-plane which therefore encloses the pole. I is given by the residue of this pole at $\alpha(\beta)$ where $\Phi(0;\,\alpha(\beta),\,\beta) = 0$:

$$I = \mathrm{Re}\left\{\frac{1}{\pi}\int_0^{\infty}\frac{\beta}{\beta^2-\omega^2}\frac{\Phi(y;\,\alpha(\beta),\beta)}{\partial\Phi\,(0;\,\alpha(\beta)\beta)/\partial\alpha}\exp\{i(\alpha(\beta)x-\beta t)\}d\beta\right\} \quad \text{for} \quad x < 0. \tag{9}$$

(*b*) The second contour is chosen to pass below the pole

$$I = \mathrm{Re}\left\{\frac{-1}{\pi}\int_0^{\infty}\frac{\beta}{\beta^2-\omega^2}\frac{\Phi(y;\,\alpha(\beta),\beta)}{\partial\Phi(0;\,\alpha(\beta),\beta)/\partial\alpha}\exp\{i(\alpha(\beta)x-\beta t)\}d\beta\right\} \quad \text{for} \quad x > 0, \tag{10}$$

and

$$I = 0 \quad \text{for} \quad x < 0. \tag{11}$$

The final solution is obtained after integrating over the β-plane. First, consider the contour (*a*) in the α-plane. (9) must be evaluated along a path just above the real axis to ensure the convergence of (4). Expanding $\alpha(\beta)$ near the real axis we get

$$I_{x<0} = \mathrm{Re}\left\{\frac{1}{\pi}\int_0^\infty \frac{\beta}{(\beta^2-\omega^2)}\frac{\Phi(y;\,\alpha(\beta),\beta)}{\partial\Phi(0;\,\alpha(\beta),\beta)/\partial\alpha}\right.$$

$$\left.\times \exp i[\{\alpha(\beta_r)+i\beta_i\, d\alpha(\beta_r)/d\beta\}x-\beta_r t-i\beta_i t]\,d\beta\right\}. \qquad (12)$$

The integration contour can be closed by a path from the origin to infinity in such a way that the exponent in (12) is negative, that is

$$\alpha_i(\beta_r)\,x+\beta_i\left(\frac{\partial\alpha_r(\beta_r)}{\partial\beta_r}x-t\right) > 0 \quad \text{for} \quad x < 0. \qquad (13)$$

Since the pole at $\alpha(\beta)$ does not lie on the imaginary axis ($\alpha_i(\beta_r) \neq 0$) we can select a value of β_i for every β_r to satisfy (13) provided $\{\partial\alpha_r(\beta_r)/\partial\beta_r\}x-t$ is not small. The case of large $\alpha_i(\beta_r)$, or small $\partial\alpha_r(\beta_r)/\partial\beta_r$ will be considered later. Now provided condition (13) holds, the contribution to the value of the integral along this path tends to zero, the asymptotic value for large t being given solely by the residue of the pole if this is enclosed by the integration circuit, that is if $\beta_i < 0$ when β_r equals ω.

Thus for $\beta_i(\omega) < 0$ we have

$$I_{x<0} \to \mathrm{Re}\left\{\frac{-i}{2}\frac{\Phi(y;\,\alpha(\omega),\omega)}{\partial\Phi(0;\,\alpha(\omega),\omega)/\partial\alpha}\exp\left[i(\alpha(\omega)\,x-\omega t)\right]\right\}, \qquad (14)$$

and for $\beta_i(\omega) > 0$, $I \to 0$.

The initial condition that no disturbance exists prior to time $t = 0$ is only satisfied for a path not enclosing the pole, that is for $\beta_i(\omega) > 0$. It follows that contour (a) in the α-plane can only be used when $\partial\alpha_r(\omega)/\partial\beta_r < 0$. Contour ($b$) allows a similar analysis to be performed, and this shows that the initial unperturbed state can only be complied with if $\partial\alpha_r(\omega)/\partial\beta_r > 0$.

The contribution to $v(y;\,x,t)$ of the remaining poles situated along the imaginary axis in the α-plane can be found by a technique similar to that used for a pole off the axis. It is not difficult to show that the path of integration in the α-plane must pass through the origin if the final solution is to comply with the initial condition of zero disturbance in the flow before the excitation starts at $t = 0$. The terms arising from poles of this type are of the form

$$\text{(i)}\ \ \alpha_i(\omega) < 0 \qquad I \sim \mathrm{Re}\{\exp[-\alpha_i(\omega)\,x-i\omega t]\} \quad (x > 0)$$

$$= 0 \quad (x < 0);$$

$$\text{(ii)}\ \ \alpha_i(\omega) < 0 \qquad I = 0 \quad (x > 0)$$

$$\sim \mathrm{Re}\{\exp[-\alpha_i(\omega)\,x-i\omega t]\} \quad (x < 0).$$

The sum of a set of terms like this will be of the form

$$\mathrm{Re}\{e^{-i\omega t}P(x)\}, \qquad (15)$$

where $P(x)$ is a function which decays for both positive and negative values of x. (15) represents the transient part of the disturbance.

Far from the source (15) will have decayed and any disturbance arises solely from the isolated pole in the α-plane which occurs through the zero in $\Phi(0; \alpha(\omega),\omega)$, and is thus related to an eigenmode of the system. For the usual case where $\partial\alpha_r(\omega)/\partial\beta_r$ is positive, the solution exists only downstream of the source and is given by:

$$v(y;\, x,t) = \mathrm{Re}\left\{\frac{i}{2}\frac{\Phi(y;\, \alpha(\omega),\omega)}{\partial\Phi(0;\, \alpha(\omega),\omega)/\partial\alpha}\exp\left[i(\alpha(\omega)x-\omega t)\right]\right\}. \tag{16}$$

In flows with $\partial\alpha_r(\omega)/\partial\beta_r$ negative the disturbance exists upstream of the point of the input and is given by (14). In boundary layer flows we can expect the amplification factor $\alpha_i(\omega)$ to be quite small compared with the wave number $\alpha_r(\omega)$, and it can therefore easily be shown, that, for a characteristic equation

$$F(\alpha,\beta) = 0,$$

we have

$$\left(\frac{\partial\beta_r}{\partial\alpha_r}\right)_{\alpha_i} = \left(\frac{\partial\alpha_r}{\partial\beta_r}\right)^{-1}_{\beta_i} + O(\alpha_{im}^2),$$

where α_{im} is the maximum value of α_i and $\partial\beta_r/\partial\alpha_r$ is the group velocity. $\partial\alpha_r/\partial\beta_r$ has the same sign as the group velocity and is approximately equal to its reciprocal.

4. Discussion

In the types of flow normally of interest the group velocity is positive and a ribbon type of disturbance therefore excites a travelling wave system downstream of the source. This mode will have a wave-number α identical to that of the free eigenmode which has the same frequency as the ribbon. Now, this wave-number will in general be complex, demonstrating that the mode excited by a vibrating ribbon is indeed one having spatial growth as observed in experiment.

Before passing on to any further discussion arising from these complex wave-number modes, it is worth examining briefly cases where the group velocity is negative, and the train of waves generated by the source propagates upstream. After sufficient time has elapsed from starting the ribbon in motion the whole upstream region will be covered with waves, and any upstream propagating disturbances would presumably effect the whole flow field by creating turbulence ahead of the source.

It has been shown that a spatially growing wave pattern is generated by a localized oscillating source, such as a ribbon, and it is therefore necessary to re-evaluate amplification rates for these modes. However, it can be shown (Gaster 1962) that for the small rates of amplification expected in boundary layers a simple transformation enables the spatial growth to be derived from the temporal growth, but for wake or jet flows it is necessary to solve the Orr–Sommerfeld equation for β real and α complex to represent satisfactorily waves of the type discussed here. A further complication arises in the study of amplified three-dimensional waves of spatially growing kind, because Squire's (1933) transformation cannot be applied directly when α is complex, and it is therefore no longer clear that these waves cannot play an important part in the transition process, even in regions of linear growth.

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